The Achilles Heel of the GSR-Shuffle *

A Note on New Age Solitaire

Anke van Zuylen Frans Schalekamp
School of Operations Research and Industrial Engineering
Cornell University
Ithaca, NY 14853
Email: anke@orie.cornell.edu; frans@cs.cornell.edu

Abstract

We show that winning the game New Age Solitaire only depends on the number of rising sequences in the deck used. The probability of winning for the special case of a new deck that is shuffled using the GSR-shuffle (and two variants) are studied. We show that this game pinpoints the Achilles heel of the GSR-shuffle as is demonstrated using the variation distance.

1 Introduction

Since the publication of [1] by Dave Bayer and Persi Diaconis, it is a commonly held belief, that shuffling a deck of 52 cards 7 times ensures a "sufficiently random" card order. In Peter Doyle's game New Age Solitaire, however, the probability of winning after 7 GSR-shuffles using a new deck is close to 80 percent, in contrast with the theoretical probability of winning of 50 percent in a uniformly random deck. This probability is known from simulation, see e.g. Brad Mann [3]; in this note, we show that this probability can actually be computed without too much difficulty. We will further show that this game points out the Achilles heel of the most commonly studied model of shuffling, the uniform riffle shuffle, also known as GSR-shuffle. The authors think the results in the first sections of this note can be used as a textbook example for undergraduates, because the mathematics is not hard, and it nicely illustrates the meaning of the variation distance.

The outline of this note is as follows: In section 2, we will give a description of the game. In section 3, we will give the definitions and some results on the GSR-shuffle, which we will use in the next sections. Section 4 contains the derivation of the probability of winning New Age Solitaire after GSR-shuffles as advertised. Section 5 demonstrates how New Age Solitaire is the Achilles heel of the GSR-shuffle. Variations of the game are studied in section 6. The proofs that are suppressed to keep the flow of the argument, can be found in the appendix.

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2 New Age Solitaire

In the game New Age Solitaire, you pass through a deck of cards held face down and make two piles, Yin and Yang. The Yin pile will consist of the suits Hearts and Clubs in the order Ace to King of Hearts followed by Ace to King of Clubs, the Yang pile will have the Diamonds and Spades suits, in order Ace to King of Spades followed by Ace to King of Diamonds. Cards can only be placed on the piles in this order. You consider the cards in the order of the deck you are given. If you cannot place a card on the Yin or the Yang pile, it is put on a third pile. You win, if the Yin pile is finished before the Yang pile. If neither of the piles is finished after all cards in the deck have been considered, you pick up the third pile, turn it over, and continue with these cards. Note that turning the pile of remaining cards implies that the order in which you consider these cards is the same as the original order of the cards.

In a new deck of cards, bought in the United States, the cards when held face down are in the order Ace to King of Hearts, Ace to King of Clubs, King to Ace of Diamonds, King to Ace of Spades. Note that the Yang pile has a disadvantage if a new deck is used; in a new deck the cards for the Yin pile are in such order that you can add them all in the first pass, whereas for the Yang pile only one card can be added in each pass through the deck.

This game can obviously be played with any deck of cards, not just with a new deck. It is interesting to study this game when a new deck is used, however, to see how an unfavorable ordering of the cards before shuffling can still be noticed even after as many as 7 shuffles. It has been shown by simulation that the probability of winning this exciting game is about 80 percent when using a new deck that has been riffle shuffled 7 times.

In one variation of the game, it is assumed that an additional cut is performed after shuffling. Our main focus in this note is on the probability of winning New Age Solitaire when using a new deck that has been riffle shuffled, and we will make a point as to why this is the most interesting situation to be studied. We will, however, also consider the probability of winning after 7 shuffles and an additional cut.

Another variation of this game, which can also be found on the internet, is the making of four, instead of two piles; one for each suit. Winning the game is then defined as finishing both the Hearts and Clubs piles, before Diamonds and Spades. We will also consider this game in section 6, where we will argue that this version of the game also constitutes a less interesting case.

3 GSR-shuffle

For ease of notation, we will assume henceforth that the cards are numbered from $1, \ldots, n$. Moreover, we will assume that the cards are in this order in a new deck, i.e. before shuffling. This implies that the Yin pile will be made by adding the cards $1, \ldots, n/2$ in this order, and the Yang pile will be made by adding the cards $n, n-1, \ldots, n/2+1$ in this order, i.e. in order which is the reverse of the numbering of the cards.

Notation We will write $\pi = (\pi_1\ \pi_2\ \cdots\ \pi_n)$ for a permutation of length n, where π_j is the number on the card in position j. Inversely, denote by $\mathrm{pos}(i, \pi)$ the position of the card numbered i in permutation π. ($\mathrm{pos}(i, \pi) = \pi_i^{-1}$, where π^{-1} is the algebraic inverse of π.) Where there is no risk of confusion, we will write $\mathrm{pos}(i)$.

The most commonly studied model of shuffling is the Uniform Riffle Shuffle, also known as the GSR-shuffle, introduced by and named after Gilbert, Shannon and Reeds [2], [4]. When performing a riffle shuffle, a deck is cut into two piles. The piles are taken in the left and right hand, and the cards from the two hands are then interleaved. The characteristic features of the GSR-model are the binomial (n, $p = 1/2$) distribution of the position of the cut, and the assumption that the probability that the next card comes from the left or right hand, is proportional to the number of cards in that hand. For more details, we would like to refer the reader to Bayer and Diaconis [1].

Bayer and Diaconis [1] showed that the probability of obtaining a certain ordering after k GSR-shuffles, only depends on the number of *rising sequences* in this ordering.

Definition A *rising sequence* is a maximal sequence of card numbers $i, i+1, \ldots, i+j$, such that $\text{pos}(i) < \text{pos}(i+1) < \cdots < \text{pos}(i+j)$.

Note that maximal implies that $\text{pos}(i-1) > \text{pos}(i)$ for $i > 1$, and $\text{pos}(i+j+1) < \text{pos}(i+j)$ for $i+j < n$.

We will also use the notion of *descending sequence*:

Definition A *descending sequence* is a maximal sequence of card numbers $i, i-1, \ldots, i-j$, such that $\text{pos}(i) < \text{pos}(i-1) < \cdots < \text{pos}(i-j)$.

We have the following cute relation between the number of rising sequences and the number of descending sequences in a permutation:

Lemma Let r be the number of rising sequences, and d be the number of descending sequences in a permutation of length n. We have the relation $r + d = n + 1$.

The proof is given in the appendix.

Proposition (Bayer & Diaconis [1]) The probability that s GSR-shuffles on a new deck result in a permutation π, is

$$P(\pi | s \text{ GSR-shuffles}) = \frac{\binom{n+2^s-r}{n}}{2^{sn}} \tag{1}$$

where r is the number of rising sequences in π.

Even though the proof gives insight in the mechanics of the GSR-shuffle, we will omit it, because it is not needed for this note.

To evaluate the "degree of randomness" after s shuffles, the probability distribution on the permutations after s shuffles is compared to the uniform probability distribution on the permutations. The usual way to compare two probability distributions is the *variation distance*, which is just one half times the L_1-norm:

$$\|Q^{*s} - U\| = \frac{1}{2} \sum_{\pi \in \mathcal{S}} \Big| P_{Q^{*s}}(\pi) - P_U(\pi) \Big| = \frac{1}{2} \sum_{\pi \in \mathcal{S}} \Big| P_{Q^{*s}}(\pi) - \frac{1}{n!} \Big|, \tag{2}$$

where Q^{*s} denotes the probability distribution on the permutations after s shuffles, U denotes the uniform probability distribution, and $\mathcal{S}$ the set of all possible permutations of length n. Note that $\|Q^{*s} - U\| = \max_{\Pi \subseteq \mathcal{S}} \sum_{\pi \in \Pi} (P_{Q^{*s}}(\pi) - P_U(\pi))$, which we will use later.

By the proposition, we can express the variation distance after s GSR-shuffles as

$$\|Q^{*s} - U\| = \frac{1}{2} \sum_{r=1}^{n} A_{n,r} \Big| \frac{\binom{n+2^s-r}{n}}{2^{sn}} - \frac{1}{n!} \Big| \tag{3}$$

where $A_{n,r}$ is the number of permutations of length n, with exactly r rising sequences. It turns out that $A_{n,r}$ are the Euler numbers, as shown by Tanny [5].

4 Winning New Age Solitaire

The main result of this note, is the fact that winning New Age Solitaire, only depends on the number of rising sequences in the deck. Henceforth, we will assume that the number of cards in the deck, n, is even.

Lemma Let r be the number of rising sequences in a deck of n cards. Then, you will win New Age Solitaire using this deck, iff $r \leq n/2$.

Proof Consider the deck restricted to cards $1, 2, \ldots, n/2$. Note that the number of rounds needed to complete the Yin pile, equals the number of rising sequences in this restricted deck, say r_1.

Similarly, the number of descending sequences in the deck restricted to cards $n/2+1, n/2+2, \ldots, n$, equals the number of rounds needed to complete the Yang pile, say d_2.

Note that you win the game, if $r_1 < d_2$, or $r_1 = d_2$ and $\text{pos}(n/2) < \text{pos}(n/2+1)$.

Denote by r_2 the number of rising sequences in the deck restricted to cards $n/2+1, n/2+2, \ldots, n$. Note that the total number of rising sequences in the deck, is equal to $r_1 + r_2 - \mathbf{1}\{\text{pos}(n/2) < \text{pos}(n/2+1)\}$, where $\mathbf{1}$ is the indicator function. By the lemma in section 3, $r_2 = n/2 + 1 - d_2$. Combining these two equalities, gives

$$r = r_1 - d_2 + n/2 + 1 - \mathbf{1}\{\text{pos}(n/2) < \text{pos}(n/2+1)\} \tag{4}$$

"$\Rightarrow$" Suppose you will win New Age Solitaire. Therefore either (1) $r_1 < d_2$, or (2) $r_1 = d_2$ and $\text{pos}(n/2) < \text{pos}(n/2+1)$. Now, using Eq. (4), we find that you will win if

(1) $r < n/2 + 1 - \mathbf{1}\{\text{pos}(n/2) < \text{pos}(n/2+1)\}$. In other words if $r \leq n/2$ when $\text{pos}(n/2) > \text{pos}(n/2+1)$, or if $r < n/2$ when $\text{pos}(n/2) < \text{pos}(n/2+1)$

(2) $r = n/2$ and $\text{pos}(n/2) < \text{pos}(n/2+1)$.

Combining (1) and (2) gives the result.

"$\Leftarrow$" Suppose $r \leq n/2$. Then Eq. (4) gives $r_1 \leq d_2 - 1 + \mathbf{1}\{\text{pos}(n/2) < \text{pos}(n/2+1)\}$, i.e. $r_1 < d_2$, or $r_1 \leq d_2$ and $\text{pos}(n/2) < \text{pos}(n/2+1)$, exactly the conditions for winning. □

By the proposition in section 3, we can now give a closed form solution for the probability of winning New Age Solitaire, using a new deck that is shuffled s times, using the GSR-shuffle.

Corollary The probability of winning New Age Solitaire, with a new deck of size n, that is GSR-shuffled s times, equals

$$\sum_{r=1}^{n/2} A_{n,r} \frac{\binom{n+2^s-r}{n}}{2^{sn}},$$

where $A_{n,r}$ are the Euler numbers.

Plugging in the numbers, yields a probability of approximately .807 of winning, after GSR-shuffling a deck of 52 cards 7 times.

5 Relation to the Variation Distance

Lemma Let Q^{*s} be the distribution on the permutations after s GSR-shuffles, and U be the uniform distribution on the permutations. For s "big enough" (Mann [3] suggests $s > 2\log_2 n$), we have

$$P(\text{winning New Age Solitaire after } s \text{ GSR-shuffles}) = \|Q^{*s} - U\| + \frac{1}{2} \tag{5}$$

Proof Since the function $\binom{n+2^s-r}{n}/2^{sn}$ is non-increasing in r, there exists a "crossover point" v for which $\binom{n+2^s-r}{n}/2^{sn} \geq 1/n!$ for all $r \leq v$ and $\binom{n+2^s-r}{n}/2^{sn} < 1/n!$ for all $r > v$.

Consequently the variation distance is equal to

$$\|Q^{*s} - U\| = \max_{\Pi \subseteq \mathcal{S}} \sum_{\pi \in \Pi} (P_{Q^{*s}}(\pi) - P_U(\pi)) = \sum_{r=1}^{v} A_{n,r} \Big(\frac{\binom{n+2^s-r}{n}}{2^{sn}} - \frac{1}{n!} \Big). \tag{6}$$

It is easy to see that the crossoverpoint v is non-decreasing in s and will move to $n/2$ for n even. Following Mann [3], v equals $n/2$ for $s > 2\log_2 n$.

So, for s "big enough" the probability of winning New Age Solitaire after s GSR-shuffles is equal to

$$\sum_{r=1}^{n/2} A_{n,r} \frac{\binom{n+2^s-r}{n}}{2^{sn}} = \sum_{r=1}^{n/2} A_{n,r} \big(\frac{\binom{n+2^s-r}{n}}{2^{sn}} - \frac{1}{n!} \big) + \frac{1}{2} = \|Q^{*s} - U\| + \frac{1}{2}. \tag{7}$$

The first equality uses the symmetry in the Euler numbers, i.e. $A_{n,r} = A_{n,n+1-r}$ and therefore $\sum_{r=1}^{n/2} A_{n,r} = \sum_{r=n/2+1}^{n} A_{n,r} = n!/2$. □

Note that it is impossible to devise a game where the difference between the probability of winning with a non-uniformly distributed deck and a uniformly distributed deck would be greater than the variation distance between the two, as can be seen by the "max set" description of the variation distance: any deterministic card game can be seen as a partitioning of the permutation space into a winning and a losing set. It is easy to see that this result can be extended to games with random strategies.

Hence, New Age Solitaire exactly pinpoints the Achilles heel of the GSR-shuffle.

6 Variations on circumstances and game rules

In this section we will consider two variations that can be found, e.g. on the internet. In the first variation the game is the same, but the deck of cards that is used is a new deck of cards has been riffle shuffled 7 times followed by an additional cut. The second variation is a variation on the rules of the game. Instead of making two piles of cards, one with the cards Ace to King of Hearts followed by Ace to King of Clubs, and one with Ace to King of Spades followed by Ace to King of Diamonds, the player now makes four piles, one for each suit. The player is said to win if the Hearts and Clubs are both finished before the Spades and Diamonds piles.

We will show that both of these variations make a less interesting object for analysis. The relationship between the probability of winning and the variation distance between the distribution after GSR-shuffles and the uniform distribution (and not so much the amusement value for the player!) is obviously the reason why this game is of interest. We will show that this relationship does not hold for the two variations, i.e. that the probability of winning for these variations is strictly less than the probability of winning for the "basic" game.

6.1 Probability of winning after s GSR-shuffles and an additional cut

The analysis of the probability of winning, when the deck is cut after the shuffles are performed, is a bit more involved. We can derive the probability of winning, however.

Notation Let π be a permutation. Denote by $\pi^{(c)}$ the permutation that is the result of cutting the permutation right after position c, and concatenating the two pieces in opposite order, i.e. for $\pi = (1\ 2\ 3\ \cdots\ n)$,

$$\pi^{(c)} = (c+1\ c+2\ \cdots\ n\ 1\ 2\ \cdots\ c).$$

Let from here on $\mathrm{rs}(\pi)$ denote the number of rising sequences of π.

The following lemma is proved in the appendix.

Lemma Let π be a permutation. Then

$$\mathrm{rs}(\pi^{(c)}) = \mathrm{rs}(\pi) + \mathbf{1}\{\mathrm{pos}(1,\pi) \leq c\} - \mathbf{1}\{\mathrm{pos}(n,\pi) \leq c\}. \tag{8}$$

Note that this implies that the number of rising sequences cannot increase nor decrease by more than 1, due to a cut.

We will now focus our attention on the GSR-shuffle. In this case we know that the probability of a permutation after shuffling only depends on the number of rising sequences of the permutation. We will derive an expression for the number of permutations with r rising sequences, in which a cut causes an increase or decrease in the number of rising sequences. Next we will use these, to calculate the probability of winning New Age Solitaire, i.e. that the deck after shuffling and an additional cut, has $n/2$ rising sequences or less.

Let's define the following numbers:

$B_{n,r,i,j} := \#$ permutations of length n, with r rising sequences, such that $\mathrm{pos}(1) = i$ and $\mathrm{pos}(n) = j$.

By the lemma above, we have for all c,

$$\#\{\pi : \mathrm{rs}(\pi) = r \text{ and } \mathrm{rs}(\pi^{(c)}) = r-1\} = \#\{\pi : \mathrm{rs}(\pi) = r \text{ and } \mathrm{pos}(1,\pi) > c \text{ and } \mathrm{pos}(n,\pi) \leq c\} \tag{9}$$

$$= \sum_{i=c+1}^{n} \sum_{j=1}^{c} B_{n,r,i,j} \tag{10}$$

and

$$\begin{aligned}\#\{\pi : \mathrm{rs}(\pi) = r \text{ and } \mathrm{rs}(\pi^{(c)}) = r+1\} &= \#\{\pi : \mathrm{rs}(\pi) = r \text{ and } \mathrm{pos}(1,\pi) \le c \text{ and } \mathrm{pos}(n,\pi) > c\} &(11)\\ &= \sum_{i=1}^{c}\sum_{j=c+1}^{n} B_{n,r,i,j}. &(12)\end{aligned}$$

It is not hard to see that we can indeed compute the numbers $B_{n,r,i,j}$ for all n, r, i, j, by setting up recursive relations. In the appendix, we will outline a way to do this.

Corollary The probability of winning New Age Solitaire, with a new deck of size n, that is GSR-shuffled s times, followed by a binomial $(n,\, p = 1/2)$ cut, equals

$$\sum_{r=1}^{n/2} \frac{\binom{n+2^s-r}{n}}{2^{sn}} A_{n,r} - \frac{\binom{n+2^s-n/2-1}{n-1}}{2^{sn}} \sum_{c=0}^{n} \binom{n}{c} \frac{1}{2^n} \sum_{i=1}^{c}\sum_{j=c+1}^{n} B_{n,n/2,i,j} < \sum_{r=1}^{n/2} \frac{\binom{n+2^s-r}{n}}{2^{sn}} A_{n,r} \qquad (13)$$

where $A_{n,r}$ are the Euler numbers (the number of permutations of length n with r rising sequences).

To see the result above, first note that you win New Age Solitaire, if the number of rising sequences after the cut is at most $n/2$. In other words, you win if the number of rising sequences before the cut is at most $n/2$, except when the number of rising sequences before the cut is $n/2$, and the cut causes an increase in the number of rising sequences, or if the number of rising sequences before the cut is $n/2+1$, and the cut causes a decrease in the number of rising sequences, i.e., let π be the permutation before the cut, and let the random variable C be the position of the cut, then

$$\begin{aligned} P(\text{winning}) &= P(\mathrm{rs}(\pi) \le n/2) &&- \sum_{c=0}^{n} P(C=c)P(\mathrm{rs}(\pi) = n/2 \text{ and } \mathrm{rs}(\pi^{(c)}) = n/2+1)\\ & &&+ \sum_{c=0}^{n} P(C=c)P(\mathrm{rs}(\pi) = n/2+1 \text{ and } \mathrm{rs}(\pi^{(c)}) = n/2)\\ &= \sum_{r=1}^{n/2} \frac{\binom{n+2^s-r}{n}}{2^{sn}} A_{n,r} &&- \frac{\binom{n+2^s-n/2}{n}}{2^{sn}} \sum_{c=0}^{n} P(C=c) \sum_{i=1}^{c}\sum_{j=c+1}^{n} B_{n,n/2,i,j}\\ & &&+ \frac{\binom{n+2^s-(n/2+1)}{n}}{2^{sn}} \sum_{c=0}^{n} P(C=c) \sum_{i=c+1}^{n}\sum_{j=1}^{c} B_{n,n/2+1,i,j}. \end{aligned} \qquad (14)$$

Now note that $B_{n,r,i,j} = B_{n,n+1-r,j,i}$. This can be seen by considering permutations $\sigma = (\sigma_1\ \sigma_2\ \cdots\ \sigma_n)$ and $\tilde{\sigma} = (n+1-\sigma_1\ \ n+1-\sigma_2\ \cdots\ n+1-\sigma_n)$, and remembering that the number of rising sequences in a permutation plus the number of descending sequences, add up to $n+1$.

Therefore, we can write

$$P(\text{winning}) = \sum_{r=1}^{n/2} \frac{\binom{n+2^s-r}{n}}{2^{sn}} A_{n,r} - \Big[\Big(\frac{\binom{n+2^s-n/2}{n}}{2^{sn}} - \frac{\binom{n+2^s-(n/2+1)}{n}}{2^{sn}}\Big) \sum_{c=0}^{n} P(C=c) \sum_{i=1}^{c}\sum_{j=c+1}^{n} B_{n,n/2,i,j}\Big]. \qquad (15)$$

Finally, since $\binom{a}{b} = \binom{a-1}{b-1} + \binom{a-1}{b}$ and thus $\binom{a}{b} - \binom{a-1}{b} = \binom{a-1}{b-1}$, this is equal to

$$P(\text{winning}) = \sum_{r=1}^{n/2} \frac{\binom{n+2^s-r}{n}}{2^{sn}} A_{n,r} - \frac{\binom{n+2^s-n/2-1}{n-1}}{2^{sn}} \sum_{c=0}^{n} P(C=c) \sum_{i=1}^{c}\sum_{j=c+1}^{n} B_{n,n/2,i,j}. \qquad (16)$$

Using the probabilities for the binomial cut, gives the equality above.

Now note that all terms, except maybe $B_{n,n/2,i,j}$, are strictly positive. Also, it is easy to see that $B_{n,n/2,1,n} = A_{n-2,n/2}$, which is strictly positive for $n = 4, 6, \ldots$. For $n = 2$, $B_{2,1,1,2} = 1$. Therefore the probability above is strictly less than $\sum_{r=1}^{n/2} \frac{\binom{n+2^s-r}{n}}{2^{sn}} A_{n,r}$ (for all n even), which is the probability of winning without the additional cut. □

Plugging in the same numbers as before, gives a probability of approximately .793 of winning, after GSR-shuffling a deck of 52 cards 7 times, followed by a binomial ($n = 52$, $p = 1/2$) cut.

6.2 New Age Solitaire with four piles

A variation on the rules of New Age Solitaire has the player make four piles instead of just two. Each pile must be made by adding the cards of one suit in the order Ace to King. In this version the player is said to win if the Hearts and Clubs piles (the Yin piles) are finished before the Diamonds and Spades piles (the Yang piles) are both finished. We will call this version of the game 4-pile Solitaire.

Lemma The probability of winning 4-pile solitaire with a deck that is shuffled using s GSR-shuffles, is strictly less than the probability of winning New Age Solitaire, with a deck that is shuffled using s GSR-shuffles.

Proof Define $W_{n,r}$ as the number of permutations with r rising sequences for which you will win 4-pile New Age Solitaire. Let $L_{n,r} = A_{n,r} - W_{n,r}$ (the number of "losing permutations"). The probability of winning the 4-pile version after s GSR-shuffles can then be expressed as follows

$$\begin{aligned}
& P(\text{winning 4-pile Solitaire after } s \text{ GSR-shuffles}) \\
= & \sum_{r=1}^{n} W_{n,r} \frac{\binom{n+2^s-r}{n}}{2^{sn}} && (17) \\
= & \sum_{r=1}^{n/2} W_{n,r} \frac{\binom{n+2^s-r}{n}}{2^{sn}} + \sum_{r=n/2+1}^{n} W_{n,r} \frac{\binom{n+2^s-r}{n}}{2^{sn}} && (18) \\
= & \sum_{r=1}^{n/2} (A_{n,r} - L_{n,r}) \frac{\binom{n+2^s-r}{n}}{2^{sn}} + \sum_{r=n/2+1}^{n} W_{n,r} \frac{\binom{n+2^s-r}{n}}{2^{sn}} && (19) \\
= & \sum_{r=1}^{n/2} A_{n,r} \frac{\binom{n+2^s-r}{n}}{2^{sn}} - \sum_{r=1}^{n/2} L_{n,r} \frac{\binom{n+2^s-r}{n}}{2^{sn}} + \sum_{r=1}^{n/2} W_{n,n+1-r} \frac{\binom{n+2^s-(n+1-r)}{n}}{2^{sn}} && (20)
\end{aligned}$$

Now note that $L_{n,r} = W_{n,n+1-r}$, since for each permutation $\sigma = (\sigma_1\ \sigma_2\ \cdots\ \sigma_n)$ with r rising sequences which is losing, there exists a permutation $\tilde{\sigma} = (n+1-\sigma_1\ \ n+1-\sigma_2\ \cdots\ n+1-\sigma_n)$, which has $n+1-r$ rising sequences, which is winning, and vice versa.

$$\begin{aligned}
(20) \quad = & \sum_{r=1}^{n/2} A_{n,r} \frac{\binom{n+2^s-r}{n}}{2^{sn}} - \sum_{r=1}^{n/2} L_{n,r} \left(\frac{\binom{n+2^s-r}{n}}{2^{sn}} - \frac{\binom{n+2^s-(n+1-r)}{n}}{2^{sn}} \right) && (21) \\
< & \sum_{r=1}^{n/2} A_{n,r} \frac{\binom{n+2^s-r}{n}}{2^{sn}}, && (22)
\end{aligned}$$

where the last inequality follows from the fact that r is at most $n/2$, and the fact that for all $n \geq 4$, there exists an $r \leq n/2$, such that $L_{n,r} > 0$: for example the permutation $\sigma = ((3/4)n\ (3/4)n-1\ \cdots\ (1/2)n+1\ n\ n-1\ \cdots\ (3/4)n+1\ 1\ 2 \cdots\ (1/2)n)$ has $n/2$ rising sequences and is losing. □

7 Conclusion

We have shown that winning the game New Age Solitaire only depends on the number of rising sequences in the deck used. Consequently, using Dave Bayer and Persi Diaconis's result [1] that the probability of a permutation after GSR-shuffles also only depends on the number of rising sequences, it is easy to compute the probability of winning with a GSR-shuffled deck. We noted that New Age Solitaire exposes exactly the weakness of the GSR-shuffle. Finally we have examined two variations, one where an additional cut is performed after the GSR-shuffles and one where the rules of the game are slightly altered. In both these cases the probability of winning is strictly less than the probability in the basic case, and therefore, besides the fact that the analysis is not as crisp as in the basic case, these cases do not form such an interesting object of study.

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9 Appendix

9.1 Counting rising sequences and descending sequences

Lemma Let r be the number of rising sequences, and d be the number of descending sequences in a permutation of length n. We have the relation $r + d = n + 1$.

Proof Look at the pair of cards number i and $i+1$, for $i = 1, \ldots, n-1$. Suppose $\text{pos}(i) < \text{pos}(i+1)$. Then cards i and $i+1$ are contained in one rising sequence, i.e. there is no rising sequence starting at card $i+1$. Also, cards i and $i+1$ are *not* contained in one descending sequence, and therefore there is a descending sequence starting at card i.

If, on the other hand, $\text{pos}(i) > \text{pos}(i+1)$, then a similar argument shows that there is a rising sequence starting with card $i+1$ and no descending sequence starting with card i.

The number of descending sequences is therefore

$$\begin{aligned} d &= \sum_{i=1}^{n} \mathbf{1}\{\text{a descending sequence starts at card number } i\} & (23)\\ &= 1 + \sum_{i=1}^{n-1} \mathbf{1}\{\text{a descending sequence starts at card number } i\} & (24)\\ &= 1 + \sum_{i=1}^{n-1} \mathbf{1}\{\text{pos}(i) < \text{pos}(i+1)\}. & (25) \end{aligned}$$

Similarly, the number of rising sequences can be expressed as

$$\begin{aligned} r &= \sum_{i=1}^{n} \mathbf{1}\{\text{a rising sequence starts at card number } i\} & (26)\\ &= 1 + \sum_{i=1}^{n-1} \mathbf{1}\{\text{a rising sequence starts at card number } i+1\} & (27)\\ &= 1 + \sum_{i=1}^{n-1} \mathbf{1}\{\text{pos}(i) > \text{pos}(i+1)\}. & (28) \end{aligned}$$

And thus $d + r = n + 1$. □

9.2 The number of rising sequences can only change by 1 after a cut

We will use another lemma to prove the lemma of section 4.

Lemma Let π be a permutation. Let $\bar{\pi}$ be the permutation that results, if the cards of the rising sequence containing n, is removed. Let $\bar{n}$ be the length of permutation $\bar{\pi}$, and let $\bar{c}$ be the position of the cut, corrected for the cards that are removed, i.e. $\bar{c} = c - \#\{i > \bar{n} : \text{pos}(i, \pi) \leq c\}$. Then

$$\text{rs}(\pi^{(c)}) = \text{rs}(\bar{\pi}^{(\bar{c})}) + \mathbf{1}\{\text{pos}(n, \pi) > c\} + \mathbf{1}\{\text{pos}(\bar{n}, \bar{\pi}) \leq \bar{c}\}, \qquad (29)$$

or equivalently

$$\text{rs}(\pi^{(c)}) = \text{rs}(\bar{\pi}^{(\bar{c})}) + \begin{cases} 0 & \text{if pos}(n,\pi) \leq c \text{ and pos}(\bar{n}, \bar{\pi}) > \bar{c} \\ 2 & \text{if pos}(n,\pi) > c \text{ and pos}(\bar{n}, \bar{\pi}) \leq \bar{c} \\ 1 & \text{otherwise.} \end{cases} \qquad (30)$$

Proof Note that since $\bar{n} + 1$ starts a new rising sequence in π, we have $\text{pos}(\bar{n} + 1, \pi) < \text{pos}(\bar{n}, \pi)$.

Also note that $\text{pos}(\bar{n}, \bar{\pi}) \leq \bar{c}$ iff $\text{pos}(\bar{n}, \pi) \leq c$.

(case 1) $\text{pos}(\bar{n}, \pi) > c$. Then the rising sequence in $\pi^{(c)}$ containing $\bar{n}$, will be the same as the rising sequence in $\bar{\pi}^{(\bar{c})}$ containing $\bar{n}$, concatenated with the part of the rising sequence in π containing $\bar{n}+1$ that is on positions 1 to c in π. Therefore, $\pi^{(c)}$ will have one more rising sequence than $\bar{\pi}^{(\bar{c})}$ only if the rising sequence containing $\bar{n} + 1$ in π, has elements that are in positions after c in π, which is equivalent to $\text{pos}(n, \pi) > c$, since n is the last element of this rising sequence in π. So we have proved

$$\text{rs}(\pi^{(c)}) = \text{rs}(\bar{\pi}^{(\bar{c})}) + \begin{cases} 0 & \text{if pos}(n,\pi) \leq c \text{ and pos}(\bar{n}, \bar{\pi}) > \bar{c} \\ 1 & \text{if pos}(n,\pi) > c \text{ and pos}(\bar{n}, \bar{\pi}) > \bar{c}. \end{cases} \qquad (31)$$

(case 2) $\text{pos}(\bar{n}, \pi) \le c$. Note that $\text{pos}(\bar{n}+1, \pi) < \text{pos}(\bar{n}, \pi) \le c$, and therefore $\bar{n}+1$ starts a new rising sequence in $\pi^{(c)}$. Moreover, if the rising sequence in π containing $\bar{n}+1$ extends beyond position c in π, the last part of the rising sequence will form a new rising sequence i n $\pi^{(c)}$. This condition is again equivalent to $\text{pos}(n, \pi) > c$. So we have proved

$$\text{rs}(\pi^{(c)}) = \text{rs}(\bar{\pi}^{(\bar{c})}) + \begin{cases} 2 & \text{if } \text{pos}(n,\pi) > c \text{ and } \text{pos}(\bar{n},\bar{\pi}) \le \bar{c} \\ 1 & \text{if } \text{pos}(n,\pi) \le c \text{ and } \text{pos}(\bar{n},\bar{\pi}) \le \bar{c}. \end{cases} \tag{32}$$

Putting the two cases together, gives the result. □

Lemma Let π be a permutation. Then

$$\text{rs}(\pi^{(c)}) = \text{rs}(\pi) + \mathbf{1}\{\text{pos}(1,\pi) \le c\} - \mathbf{1}\{\text{pos}(n,\pi) \le c\}, \tag{33}$$

or equivalently

$$\text{rs}(\pi^{(c)}) = \text{rs}(\pi) + \begin{cases} +1 & \text{if } \text{pos}(1,\pi) \le c \text{ and } \text{pos}(n,\pi) > c \\ -1 & \text{if } \text{pos}(1,\pi) > c \text{ and } \text{pos}(n,\pi) \le c \\ +0 & \text{otherwise.} \end{cases} \tag{34}$$

Proof By induction of the number of rising sequences. For π with 1 rising sequence, i.e. $\pi = (1\ 2\ \cdots\ n)$, we get $\pi^{(c)} = (c+1\ c+2\ \cdots\ n\ 1\ 2\ \cdots\ c)$ and immediately see that $\text{rs}(\pi^{(c)}) = 1$ if $c = 0$ or $c = n$, and $\text{rs}(\pi^{(c)}) = 2$ if $c = 2, \ldots, n-1$. Since $\text{pos}(1,\pi) = 1$ and $\text{pos}(n,\pi) = n$ this is exactly in accordance with the claim.

Suppose now that the claim is true for all permutations with $r-1$ rising sequences, and consider a permutation with r rising sequences, say π. Let as in the lemma before, $\bar{\pi}$ be the permutation that results, if the cards of the rising sequence containing n, is removed. Also, let $\bar{n}$ and $\bar{c}$ be as defined in this last claim.

Now, by the last claim, we have

$$\begin{aligned}
\text{rs}(\pi^{(c)}) &= \text{rs}(\bar{\pi}^{(\bar{c})}) + \mathbf{1}\{\text{pos}(n,\pi) > c\} + \mathbf{1}\{\text{pos}(\bar{n},\bar{\pi}) \le \bar{c}\} && (35) \\
&= \text{rs}(\bar{\pi}^{(\bar{c})}) + 1 - \mathbf{1}\{\text{pos}(n,\pi) \le c\} + \mathbf{1}\{\text{pos}(\bar{n},\bar{\pi}) \le \bar{c}\} && (36) \\
&= \text{rs}(\bar{\pi}) + \mathbf{1}\{\text{pos}(1,\bar{\pi}) \le \bar{c}\} - \mathbf{1}\{\text{pos}(\bar{n},\bar{\pi}) \le \bar{c}\} \\
&\quad +1 - \mathbf{1}\{\text{pos}(n,\pi) \le c\} + \mathbf{1}\{\text{pos}(\bar{n},\bar{\pi}) \le \bar{c}\} \qquad \text{(by the inductive hypothesis)} && (37) \\
&= \text{rs}(\bar{\pi}) + 1 + \mathbf{1}\{\text{pos}(1,\bar{\pi}) \le \bar{c}\} - \mathbf{1}\{\text{pos}(n,\pi) \le c\} && (38) \\
&= \text{rs}(\bar{\pi}) + 1 + \mathbf{1}\{\text{pos}(1,\pi) \le c\} - \mathbf{1}\{\text{pos}(n,\pi) \le c\} && (39) \\
&= \text{rs}(\pi) + \mathbf{1}\{\text{pos}(1,\pi) \le c\} - \mathbf{1}\{\text{pos}(n,\pi) \le c\} \qquad \text{(by the definition of } \bar{\pi}) && (40)
\end{aligned}$$

□

9.3 Recursive relations for $B_{n,r,i,j}$

We can find the following relations by deleting element 1, and conditioning on pos(1) in the shorter permutation (i.e. pos(2) in the original permutation).

For $i < j$:

$$B_{n,r,i,j} = \sum_{k=1}^{i-1} B_{n-1,r-1,k,j-1} + \sum_{k=i}^{j-2} B_{n-1,r,k,j-1} + \sum_{k=j}^{n-1} B_{n-1,r,k,j-1}. \tag{41}$$

For $i > j$:

$$B_{n,r,i,j} = \sum_{k=1}^{j-1} B_{n-1,r-1,k,j} + \sum_{k=j+1}^{i-1} B_{n-1,r-1,k,j} + \sum_{k=i}^{n-1} B_{n-1,r,k,j} \tag{42}$$

Alternatively, one could delete element n and condition on $\text{pos}(n-1)$, et cetera. Note that this recursion scheme, together with $B_{1,1,1,1} = 1$, $B_{1,r,i,j} = 0$ for $r \neq 1$, $i \neq 1$ or $j \neq 1$, indeed determines the value of $B_{n,r,i,j}$ for all n, r, i, j, since $B_{n,r,i,j}$ is only defined in terms of $B_{n-1,r',i',j'}$.